

# Remarks about the thermodynamic limit in selfgravitating systems

L. Velazquez<sup>1,\*</sup> and F. Guzman<sup>2</sup>

<sup>1</sup>*Departamento de Física, Universidad de Pinar del Río,  
Martí 270, Esq. 27 de Noviembre, Pinar del Río, Cuba.*

<sup>2</sup>*Departamento de Física Nuclear, Instituto Superior de Ciencias y  
Tecnología Nucleares, Carlos III y Luaces, Plaza, La Habana, Cuba.*

The present effort addresses the question about the existence of a well-defined thermodynamic limit for the astrophysical systems with the following *power law form*: to tend the number of particles,  $N$ , the total energy,  $E$ , and the characteristic linear dimension of the system,  $L$ , to infinity, keeping constant  $E/N^{\Lambda_E}$  and  $L/N^{\Lambda_L}$ , being  $\Lambda_E$  and  $\Lambda_L$  certain scaling exponent constant. This study is carried out for a system constituted by a non-rotating fluid under the influence of its own Newtonian gravitational interaction. The analysis yields that a thermodynamic limit of the above form will only appear when the local pressure depends on the energy density of fluid as  $p = \gamma\epsilon$ , being  $\gamma$  certain constant. Therefore, a thermodynamic limit with a power law form can be only satisfied by a reduced set of models, such as the selfgravitating gas of fermions and the Antonov isothermal model.

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In a series of recent papers, de Vega, Sanchez and Laliena discuss about the existence of a well-defined thermodynamic limit for the self-gravitating systems [1, 2, 3, 4]. The first two authors claimed that the thermodynamic limit of a self-gravitating system can be taken by letting the number of particles,  $N$ , and the volume,  $V$ , tend to infinity keeping the ratio  $N/V^{\frac{1}{3}}$  constant. On the other hand, Laliena exposes a series of reasons which in his viewpoint makes invalid the consideration of the above thermodynamic limit. The present paper is aimed to contribute to clarify the question about the existence of a well-defined thermodynamic limit for self-gravitating systems.

Our analysis starts from the consideration of a mean field approximation for the estimation of the microcanonical volume  $W$  of a non-rotating fluid under the influence of its own Newtonian gravitational interaction:

$$W_{MF}(E, N, L) = \int \mathcal{D}\rho(\mathbf{r}) \mathcal{D}\epsilon(\mathbf{r}) \mathcal{D}\phi(\mathbf{r}) \delta(E - H[\epsilon, \rho, \phi]) \delta(N - N[\rho]) \delta\{\phi(\mathbf{r}) - \mathcal{G}[\rho, \mathbf{r}]\} \exp\{S[\epsilon, \rho]\}, \quad (1)$$

being  $\rho(\mathbf{r})$  and  $\epsilon(\mathbf{r})$ , the particles and the internal energy density of the fluid at the point  $\mathbf{r}$ , and  $\phi(\mathbf{r})$ , the Newtonian potential at the same point. The functionals:

$$S[\epsilon, \rho] = \int d^3\mathbf{r} s\{\epsilon(\mathbf{r}), \rho(\mathbf{r})\}, \quad H[\epsilon, \rho, \phi] = \int d^3\mathbf{r} \epsilon(\mathbf{r}) + \frac{1}{2}m\rho(\mathbf{r})\phi(\mathbf{r}), \quad N[\rho] = \int d^3\mathbf{r} \rho(\mathbf{r}), \quad (2)$$

represent the total entropy, energy and particles number for a given profile with  $\epsilon(\mathbf{r})$  and  $\rho(\mathbf{r})$ ; and the functional  $\mathcal{G}[\rho; \mathbf{r}]$ :

$$\mathcal{G}[\rho; \mathbf{r}] = - \int \frac{Gm\rho(\mathbf{r}_1) d^3\mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|}, \quad (3)$$

is the Green solution of the Poisson problem:

$$\Delta\phi = 4\pi Gm\rho. \quad (4)$$

This approximation was analyzed in details in ref.[5]. This system has been enclosed in a spherical rigid container with characteristic linear dimension  $L$  in order to avoid the long-range divergence of the Newtonian gravitational potential. On the other hand, the short-range divergence of this interaction has been also avoided by the consideration of quantum effects or the nature size of the particles which are compose the fluid, which are taken in an implicit manner in the entropy density of the fluid,  $s\{\epsilon(\mathbf{r}), \rho(\mathbf{r})\}$ .

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\*Electronic address: luisberis@geo.upr.edu.cu

$W_{MF}(E, N, L)$  can be rewritten by using the Fourier representation of the delta functions:  $\delta(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \exp(ikx)$ , being  $z = \varepsilon + ik$ , yielding:

$$W_{MF}(E, N) \sim \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dkd\eta}{(2\pi)^2} \int \mathcal{D}\rho(\mathbf{r}) \mathcal{D}\epsilon(\mathbf{r}) \mathcal{D}\phi(\mathbf{r}) \mathcal{D}h(\mathbf{r}) \exp\{\mathcal{L}[\epsilon, \rho, \phi; z, z_1, J]\}, \quad (5)$$

where  $z = \beta + ik$  and  $z_1 = \mu + i\eta$  with  $\beta, \eta \in \mathbb{R}$ , being the functional  $\mathcal{L}[\epsilon, \rho, \phi; z, z_1, J]$  defined by:

$$\mathcal{L}[\epsilon, \rho, \phi; z, z_1, J] = S[\epsilon, \rho] + z(E - H[\epsilon, \rho, \phi]) + z_1(N - N[\rho]) + J * (\phi - \mathcal{G}[\rho]). \quad (6)$$

The functional term

$$J * (\phi - \mathcal{G}[\rho]) \equiv \int d^3\mathbf{r} J(\mathbf{r}) \{\phi(\mathbf{r}) - \mathcal{G}[\rho; \mathbf{r}]\}$$

appears as consequence of the Fourier representation of the delta functional  $\delta\{\phi(\mathbf{r}) - \mathcal{G}[\rho; \mathbf{r}]\}$ . Here,  $J(\mathbf{r})$  is a complex function,  $J(\mathbf{r}) = j(\mathbf{r}) + ih(\mathbf{r})$ , with  $j(\mathbf{r}) \in \mathbb{R}$ .

The direct integration of the functional integrals in the expression (5) is a formidable task which could be only carried out by using adequate approximations. The usual methodology which is applied in this situations is the called *steepest decent method*. The application of this method is based on the asymptotic behavior of the thermodynamical variables and potentials in the many particle limit  $N \gg 1$ , which leads to estimate that the main contribution for the expectation values for the physical observables in the microcanonical ensemble comes from the most probably configurations of the system. Thus, the Boltzmann entropy of the system,  $S_B = \ln W$ , is obtained by using a min-max procedure:

$$S_B(E, N, L) \simeq \min_{\beta, \mu, j} \left\{ \max_{\epsilon, \rho, \phi} \mathcal{L}[\epsilon, \rho, \phi; \beta, \mu, j] \right\}, \quad (7)$$

whose the stationary conditions leads to the structure equations of the equilibrium configurations. According to the definition of the functional  $\mathcal{L}[\epsilon, \rho, \phi; \beta, \mu, j]$ , the equation (6), the reader may recognize the Lagrange formalism of the Classical Thermodynamics.

The presence of an additive kinetic part in the Hamiltonian of certain system leads to an exponential growing of the microcanonical volume  $W$  with the  $N$  increasing, and therefore, the Boltzmann entropy will grow proportional to  $N$  in the many particles limit,  $S_B = \ln W \propto N$ . Therefore, let us now concentrate our attention in the analysis of the  $N$ -behavior of the thermodynamical variables with the growing of the number of particles.

The usual thermodynamic limit for the extensive systems:

$$N \rightarrow \infty, \text{ keeping constant } \frac{E}{N} \text{ and } \frac{N}{V}, \quad (8)$$

where  $V$  is the volume of the system, is directly related with the *extensive properties* of these systems when the thermodynamical variables of the system are scaled by some scaling parameter  $\alpha$  as follows:

$$\left. \begin{array}{l} N \rightarrow N(\alpha) = \alpha N, \\ E \rightarrow E(\alpha) = \alpha E, \\ V \rightarrow V(\alpha) = \alpha V, \end{array} \right\} \Rightarrow W \rightarrow W(\alpha) = \exp(\alpha \ln W). \quad (9)$$

In analogy with the extensive properties of the traditional systems, we will analyze the necessary conditions for the existence of the following *power law self-similarity* scaling behavior of the microcanonical variables  $E$ ,  $N$  and  $L$  for the selfgravitating systems:

$$\left. \begin{array}{l} N \rightarrow N(\alpha) = \alpha N, \\ E \rightarrow E(\alpha) = \alpha^{\Lambda_E} E, \\ L \rightarrow L(\alpha) = \alpha^{\Lambda_L} L, \end{array} \right\} \Rightarrow W \rightarrow W(\alpha) = \exp(\alpha \ln W), \quad (10)$$

where  $\Lambda_E$  and  $\Lambda_L$  are certain constant scaling exponent which lead to an *extensive* character of the Boltzmann entropy. This kind of self-similarity behavior is directly related with a thermodynamic limit with a power law form:

$$N \rightarrow \infty, \text{ keeping constant } \frac{E}{N^{\Lambda_E}} \text{ and } \frac{L}{N^{\Lambda_L}}. \quad (11)$$

The existence of this kind of self-similarity condition allows a considerable simplification of the thermodynamical description: the study can be performed by setting  $N = 1$  and considering the  $N$ -dependence in the scaling laws by taking  $\alpha = N$ . This scaling behavior is very useful in numerical experiments, since it allows us to extend the results of this kind of study on a finite system to much bigger systems. Contrary, the nontrivial  $N$ -dependent behavior of the thermodynamical variables and potentials leads to a complication of any kind of study.

In order to satisfy this scaling behavior for the global variables  $E$ ,  $N$ ,  $L$  and the Boltzmann entropy  $S_B$ , the local functions  $\epsilon(\mathbf{r})$ ,  $\rho(\mathbf{r})$ ,  $\phi(\mathbf{r})$  and  $s(\epsilon, \rho; \phi)$  should be scaled as follows:

$$\rho \rightarrow \rho(\alpha) = \alpha^{\Lambda_\rho} \rho \Rightarrow \left\{ \begin{array}{l} \phi \rightarrow \phi(\alpha) = \alpha^{\Lambda_\phi} \phi \\ \epsilon \rightarrow \epsilon(\alpha) = \alpha^{\Lambda_\epsilon} \epsilon \\ s \rightarrow s(\alpha) = \alpha^{\Lambda_s} s \end{array} \right\}. \quad (12)$$

Since the characteristic particles density behaves as  $\rho_c \sim N/L^3$ , the scaling exponent for the particles density is  $\Lambda_\rho = 1 - 3\Lambda_L$ . From the expression of the Newtonian potential (3) is derived that its characteristic unit is  $\phi_c \sim \rho_c L^2$ , and therefore,  $\Lambda_\phi = 1 - \Lambda_L$ . The energy scaling exponent is equal to the scaling exponent of the total gravitational potential energy, so that,  $\Lambda_E = 2 - \Lambda_L$ . The other scaling exponents are obtained by using identical reasonings. All these scaling exponents depend on the scaling exponent  $\Lambda_L$  as follows:

$$\begin{aligned} \Lambda_\rho &= 1 - 3\Lambda_L = \Lambda_s, \quad \Lambda_\phi = 1 - \Lambda_L, \\ \Lambda_\epsilon &= 2 - 4\Lambda_L, \quad \Lambda_E = 2 - \Lambda_L. \end{aligned} \quad (13)$$

In order to satisfy these scaling laws is also *necessary* that the entropy density exhibits to the following scaling behavior:

$$s(\alpha^{\Lambda_\epsilon} \epsilon, \alpha^{\Lambda_\rho} \rho) = \alpha^{\Lambda_s} s(\epsilon, \rho), \quad (14)$$

This scaling property is satisfy if  $s(\epsilon, \rho)$  obeys to the following functional form:

$$s(\epsilon, \rho) = \rho F(\epsilon/\rho^\eta), \quad (15)$$

where  $\eta = \Lambda_\epsilon/\Lambda_\rho$ , being  $F(x)$  an arbitrary function. This functional form leads to a simple relation between the local pressure  $p$  and the internal energy density  $\epsilon$  of the fluid. By taking into account the functional form (15) and using the relations

$$\beta = \frac{\partial}{\partial \epsilon} s(\epsilon, \rho), \quad \beta p = s(\epsilon, \rho) - \epsilon \frac{\partial}{\partial \epsilon} s(\epsilon, \rho) - \rho \frac{\partial}{\partial \rho} s(\epsilon, \rho), \quad (16)$$

it is straightforward derived the relation:

$$p = \gamma \epsilon, \quad (17)$$

where  $\gamma = \eta - 1$ . There are some well-known Hamiltonian systems which satisfy this kind of relation, as example, the system of nonrelativistic or ultrarelativistic noninteracting particles, without matter if they obey to the Boltzmann, Fermi-Dirac or Bose-Einstein Statistics. The scaling parameter  $\Lambda_L$  and  $\Lambda_E$  are obtained from the parameter  $\gamma$  as follows:

$$\Lambda_L = \frac{\gamma - 1}{3\gamma - 1}, \quad \Lambda_E = \frac{5\gamma - 1}{3\gamma - 1}. \quad (18)$$

This result evidences that the power laws form for the self-similarity conditions (10) can be only satisfied by a reduced group of models whose microscopic picture obeys to the relation (17). Since  $\gamma = \frac{2}{3}$  for the ideal gas of particles, the scaling exponents for the Antonov problem [7] and the selfgravitating fermions model [9, 10] are given by  $\Lambda_E = \frac{7}{3}$  and  $\Lambda_L = -\frac{1}{3}$ , and therefore, they obey to the following thermodynamic limit:

$$N \rightarrow \infty, \text{ keeping constant } \frac{E}{N^{\frac{7}{3}}} \text{ and } LN^{\frac{1}{3}}. \quad (19)$$

This thermodynamic limit was established in the ref.[8] for the self-gravitating nonrelativistic fermions by using other reasonings. Since the selfgravitating relativistic gas and classic hard sphere model [11, 12] do not obey to the relation (17), *they do not posses a scaling behavior with a power law form* (10).

This result clarifies a question about the discussion of the adequate thermodynamic limit for selfgravitating systems. All above selfgravitating models converge in the low density limit in the selfgravitating gas, and therefore, they exhibit a  $N$ -growing of the energy with the 7/3 power law. However, they diverge in regard to the high density limit since them used different regularization procedure for the short-range divergence of the Newtonian potential. Generally speaking, such power law thermodynamic limit (11) does not exist in selfgravitating systems for the whole values of the microcanonical variables. Even the selfgravitating gas of fermions exhibits this behavior only in the nonrelativistic limit since the relation (15) disappears when the relativistic effects are taken into account for a massive enough selfgravitating system.

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